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APPLICATION OF SCHMIDT'S METHOD TO THE CALCULATION OF SPALDING'S FUNCTION AND OF THE SKIN-FRICTION COEFFICIENT IN TURBULENT FLOW

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Abstract—The paper describes the details of the numerical computation of the so-called Spalding function which occurs in the theory of heat transfer across turbulent boundary layers. The present calculation is confined to the case when the Prandtl number $Pr = 1$ and when the assumption of a constant turbulent Prandtl number equal to unity can be made. The method used is a modification of Schmidt's numerical scheme for the integration of the time-dependent, one-dimensional Fourier equation of heat conduction. In the present case, the term which plays the part of thermal diffusivity depends on the analog of the space-co-ordinate and renders the equation singular at the origin. The singularity is circumvented by making use of the fact that the temperature profiles must possess zero curvature at the wall. By the use of the Reynolds analogy for turbulent boundary layers it is shown that the preceding calculations can be used to obtain a relation between the skin-friction coefficient and the length Reynolds number for a flat plate wetted by a turbulent boundary layer. The integration is very simple and provides a method of judging the accuracy of the theory. This turns out to be reasonably satisfactory.

1. STATEMENT OF PROBLEM

IN A remarkable paper on heat transfer across a turbulent boundary layer, Spalding [1, 2] showed that the problem is reducible to the following partial differential equation

$$\frac{\partial \theta}{\partial x^+} = \frac{1}{\epsilon^+ u^+} \cdot \frac{\partial}{\partial u^+}\left(\frac{1}{Pr_e}\frac{\partial \theta}{\partial u^+}\right). \tag{1}$$

The dimensionless, independent variables x^+ and u^+ are defined as

$$x^+ = \int_{x_0}^{x} \frac{v_*(x)}{\nu}\,\mathrm{d}x \tag{1a}$$

and

$$u^+ = \bar{u}/v_* \tag{1b}$$

where $v_* = (\tau_w/\rho)^{1/2}$ is the friction velocity, ν is the kinematic viscosity, $\bar{u}$ is the average longitudinal velocity component in the turbulent boundary layer, and x_0 is the position at which the thermal boundary layer begins to develop. The effective Prandtl number is given by

$$\frac{1}{Pr_e} = \frac{(1/Pr) + (\mu_t/\mu)\,.\,(1/Pr_t)}{1 + (\mu_t/\mu)} \tag{1c}$$

where Pr is the molecular Prandtl number, μ is the molecular viscosity, Pr_t is the turbulent Prandtl number, and μ_t is the eddy viscosity. For $Pr_t = Pr = 1$, we have

$$Pr_e = 1, \tag{1d}$$

and in this particular case, equation (1) assumes the simplified form

$$\frac{\partial \theta}{\partial x^+} = \frac{1}{\epsilon^+ u^+}\,\frac{\partial^2 \theta}{\partial (u^+)^2}. \tag{2}$$

The dependent variable θ represents a dimensionless form of the temperature T, in the boundary layer, given by

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{2a}$$

where T_w and T_∞ are the constant wall temperature and the constant free-stream temperature,

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respectively. Hence, for a typical heat transfer problem, equations (1) and (2) are subject to the boundary conditions

$$\left.\begin{array}{l}\theta = 0 \text{ at } x = x_0 \text{ or } x^+ = 0 \text{ and all } u^+ > 0 \\ \theta = 0 \text{ at } u^+ = \infty \text{ and all } x > x_0 \text{ or } x^+ > 0 \\ \theta = 1 \text{ at } u^+ = 0 \text{ and all } x > x_0 \text{ or } x^+ > 0\end{array}\right\}. \quad (3)$$

It is noted that the reduced velocity u^+ rather than the distance y (or its dimensionless form $y^+ = yv_*/\nu$) is used, but it is clear that $u^+ = \infty$ for $y = y^+ = \infty$ and $u^+ = 0$ for $y = y^+ = 0$ for the normally employed universal velocity profile (Coles' [3] "law of the wall"). The quantity

$$\epsilon^+ = \frac{\mu_e}{\mu} = 1 + \frac{\mu_t}{\mu}$$

denotes the effective viscosity in a turbulent boundary layer. It has been shown [1, 2] that the effective viscosity ϵ^+ can be calculated from the law of the wall by simple differentiation, since

$$\epsilon^+ = \frac{dy^+}{du^+}.$$

From the form of equations (1), (2) and (4) it is evident that it is advantageous to invert the usual expression $u^+(y^+)$ for the law of the wall, and to represent y^+ as an explicit function of u^+. An appropriate form was indicated by Spalding [4] (see also [1, 2]) who formulated the law of the wall by the equation

$$y^+(u^+) = u^+ + A\,\{\exp(\kappa u^+) - 1 - \kappa u^+ - \tfrac{1}{2}(\kappa u^+)^2 - \tfrac{1}{6}(\kappa u^+)^3 - \tfrac{1}{24}(\kappa u^+)^4\} \quad (5)$$

so that

$$\epsilon^+(u^+) = 1 + A\kappa\,\{\exp(\kappa u^+) - 1 - \kappa u^+ - \tfrac{1}{2}(\kappa u^+)^2 - \tfrac{1}{6}(\kappa u^+)^3\}. \quad (5a)$$

The proposed values for the constants were

$$\left.\begin{array}{l}A = 0{\cdot}1108 \\ \kappa = 0{\cdot}4 \\ A\kappa = 0{\cdot}04432\end{array}\right\}. \quad (5b)$$

It is convenient to introduce the abbreviation

$$f(u^+) = \epsilon^+ u^+ \quad (6)$$

which changes the form of equation (2) to

$$\frac{\partial\theta}{\partial x^+} = \frac{1}{f(u^+)}\,\frac{\partial^2\theta}{\partial(u^+)^2}. \quad (7)$$

The preceding equation, subject to the boundary conditions (3), was solved approximately by Spalding [1] by the use of the energy integral equation, and by Murali Dharan [5], who employed an analog computer for the purpose. The present note describes the method used to integrate the equation on a digital computer (the Brown University IBM 7070).

2. EXACT NUMERICAL SOLUTION

A glance at equation (7) reveals that it is related to the heat conduction equation and that it lends itself to being solved by the application of Schmidt's [6] finite difference scheme. Referring to the grid shown in Fig. 1, we can write the difference equation in the form

$$\theta(x^+ + \Delta x^+, u^+) = \theta(x^+, u^+) + \varphi[\theta(x^+, u^+ + \Delta u^+) + \theta(x^+, u^+ - \Delta u^+) - 2\theta(x^+, u^+)] \quad (7)$$

where

$$\varphi = \frac{\Delta x^+}{f(u^+)\,.\,(\Delta u^+)^2}. \quad (7a)$$

The difference equation (7) permits us to calculate the temperature profile at $x^+ + \Delta x^+$ from the profile supposed to be known at x^+. For example θ_{23} in Fig. 1 is seen to be determined from θ_{12}, θ_{13} and θ_{14} by the use of the relation

$$\theta_{23} = \theta_{13} + \varphi_3(\theta_{12} + \theta_{14} - 2\theta_{13}) \quad (7b)$$

where φ_3 is calculated from equation (7a) with $f(u^+) = f_3$, it being evident that $f(u^+)$ is constant on lines $u^+ =$ const. It is recalled that the stability of the calculation demands that the increments Δx^+ and Δu^+ must be chosen in a manner to ensure that the condition

$$\varphi = \frac{\Delta x^+}{f(u^+)\,.\,(\Delta u^+)^2} < 0{\cdot}5 \quad (7c)$$

should be satisfied at every step. Thus, if reasonably small steps Δu^+ are chosen, it becomes necessary to choose extremely small steps in Δx^+ which is practicable only if a very fast digital computer is available.

It must not be thought, however, that the preceding scheme can be carried out in a straightforward manner. Since $f(u^+) = 0$ at $u^+ = 0$, equation (7) turns out to be singular along $u^+ = 0$. This has the effect of rendering $\varphi \to \infty$ as

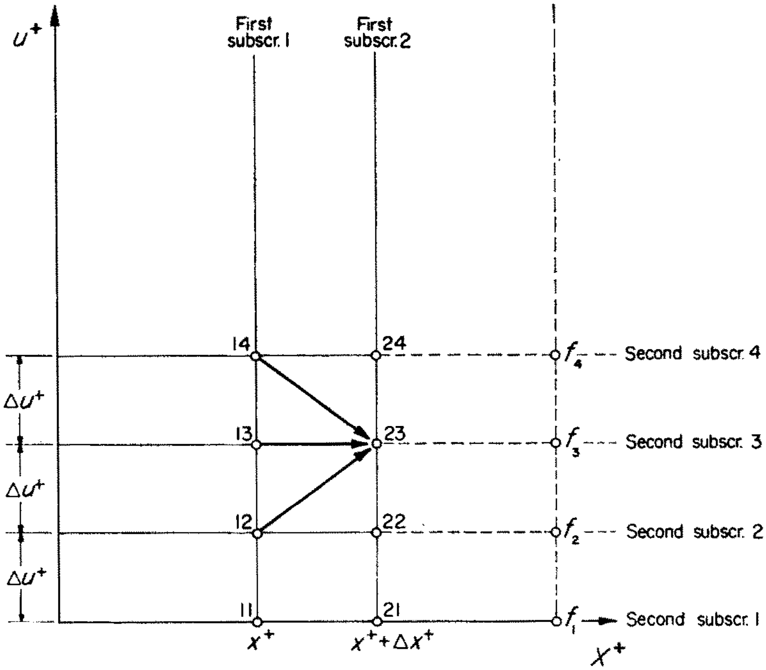

FIG. 1. Grid for the finite difference scheme.

$u^+ \to 0$, which precludes the use of equation (7) for small values of u^+ owing to the fact that $1/f(u^+)$ becomes very large. This makes it impossible to satisfy condition (7c), regardless of how small the steps Δx^+ are made. Furthermore, bearing in mind that the real aim of the calculation is to obtain the slope

$$Sp(x^+, Pr) = -\left(\frac{\partial \theta}{\partial u^+}\right)_{u^+=0}$$

at $u^+ = 0$, no compromise is possible, since large errors would be introduced precisely along that line ($u^+ = 0$) where the slope is to be evaluated. It is this slope which determines the rate of heat transfer, since as shown previously [1, 2], it relates the local Stanton number to the local skin friction coefficient, c_f, through the equation

$$St = Sp \,.\, (\tfrac{1}{2}c_f)^{1/2}. \tag{8a}$$

The so-called Spalding function $Sp(x^+, Pr)$ depends on the Prandtl number, and the present note is restricted to the calculation of $Sp(x^+, 1)$ which will be denoted by Sp_1 for short. A similar scheme can be used for Prandtl numbers different from unity, but we shall refrain from discussing this point here.

The preceding deficiency can be remedied if it is noted that the curvature of the temperature profiles is zero everywhere along the wall. This is a general characteristic of all temperature profiles along constant-temperature walls in boundary layers. Since at the wall

$$\frac{\partial \theta}{\partial x^+} = 0, \text{ and } f = 0,$$

it is easy to see that

$$\frac{\partial^2 \theta}{(\partial u^+)^2} = f \,.\, \frac{\partial \theta}{\partial x^+} = 0 \text{ at the wall.}$$

This means that the calculation must be started at "infinity", proceeding down the line $x^+ + \Delta x^+$ as far as θ_{23}. To the degree of approximation used, it is then evident that

$$\theta_{22} = \frac{1 + \theta_{23}}{2}$$

since $\theta_{21} = 1$. By these means the condition $\theta = 0$ at $u^+ = \infty$ will be satisfied automatically, and in addition the calculation will yield $\theta = 1$ and $\partial^2\theta/\partial u^+ = 0$ at $u^+ = 0$.

In principle, it is possible to start the calculation at $x^+ = 0$ with $\theta = 0$ for all values of u^+, except that $\theta = 1$ at $u^+ = 0$. However, this would put a great strain on the accuracy of the numerical scheme, and would necessitate taking very small steps in both directions. It is, therefore, more expedient to utilize the fact that Lighthill's analytic solution for laminar boundary layers [1, 2, 7, 8] provides an excellent approximation for small values of x^+. It has been shown, then, that

$$\theta(u^+, x^+) = 1 - \frac{\gamma(\frac{1}{3}, \eta)}{\Gamma(\frac{1}{3})} \tag{9}$$

where $\gamma(\frac{1}{3}, \eta)$ is the incomplete gamma function of order $\frac{1}{3}$ and of the similarity parameter

$$\eta = \frac{(y^+)^3}{9x^+} \tag{9a}$$

in which y^+ must be expressed in terms of u^+ by the use of equation (5). Furthermore, in that range

$$Sp(x^+, 1) = 0{\cdot}53835\,(x^+)^{-1/3}. \tag{9b}$$

This fact can be utilized in two ways. First, the solution can be started at some $x^+ > 0$. In our case the choice of the starting point at $x^+ = 10$ was made. Secondly, the accuracy of the numerical scheme can be tested by using it to perform a calculation below this value, i.e. in a range where the approximations in equations (9) and (9b) are still very good. In the present case, the trial calculation was performed from $x^+ = 2$ to $x^+ = 10$.

3. DETAILS OF THE SOLUTION AND RESULTS

The universal temperature profile from equation (9) was computed by matching automatically the series expansion for $\gamma(\frac{1}{3}, \eta)$,* namely

$$\gamma(\tfrac{1}{3}, \eta) = 3\eta^{1/3}\left\{1 - \frac{1}{1!4}\eta + \frac{1}{2!7}\eta^2 + \frac{1}{3!10}\eta^3 - \ldots\right\}$$

* See Ref. 9.

to its asymptotic expansion

$$\gamma(\tfrac{1}{3}, \eta) \sim \Gamma(\tfrac{1}{3}) - (\exp - \eta)\,\eta^{-2/3}\left\{1 + \sum_{n=1}^{\infty} (-\tfrac{2}{3})(-\tfrac{5}{3})\ldots(\tfrac{1}{3} - n)\,\eta^n\right\}.$$

The latter was used at $\eta \geq 7{\cdot}2$, whereas the former was employed for $\eta < 7{\cdot}2$. Here

$$\Gamma(\tfrac{1}{3}) = 2{\cdot}6789385.$$

As seen from equation (9a), the calculation of profiles for different values of x^+ merely requires the scaling of η with the factor $9x^+$ to find the corresponding y^+ and hence u^+.

The trial calculation was performed with the fixed steps

$$\Delta x^+ = 0{\cdot}02 \qquad \Delta u^+ = 0{\cdot}4 \tag{10}$$

for which the highest value of φ was

$$\varphi = 0{\cdot}16.$$

The calculation was begun at $x^+ = 2$ with Lighthill's solution, equation (8), and terminated at $x^+ = 10$, when it was compared with the analytic solution at $x^+ = 10$. The comparison showed that

$Sp_1(10) = 0{\cdot}24955737$ for the numerical solution
$Sp_1(10) = 0{\cdot}24984285$ for the analytic solution

or a discrepancy of less than 1 in 1000. A comparison of the velocity profiles is shown below:

$u^+=$	Numerical	Analytic
0·4	0·9002	0·9001
0·8	0·8004	0·8004
1·2	0·7014	0·7016
1·6	0·6043	0·6047
2·0	0·5104	0·5110

These results were considered satisfactory, since the discrepancies would be much less for $x^+ > 10$ than for $2 < x^+ < 10$ with the same steps.

A further check was made by performing the calculation from $x^+ = 10$ to $x^+ = 50$ with the steps given in (10) and with steps of half that size. The comparison is as follows:

$Sp_1(50) = 0{\cdot}14808$ full steps
$Sp_1(50) = 0{\cdot}14815$ halved steps

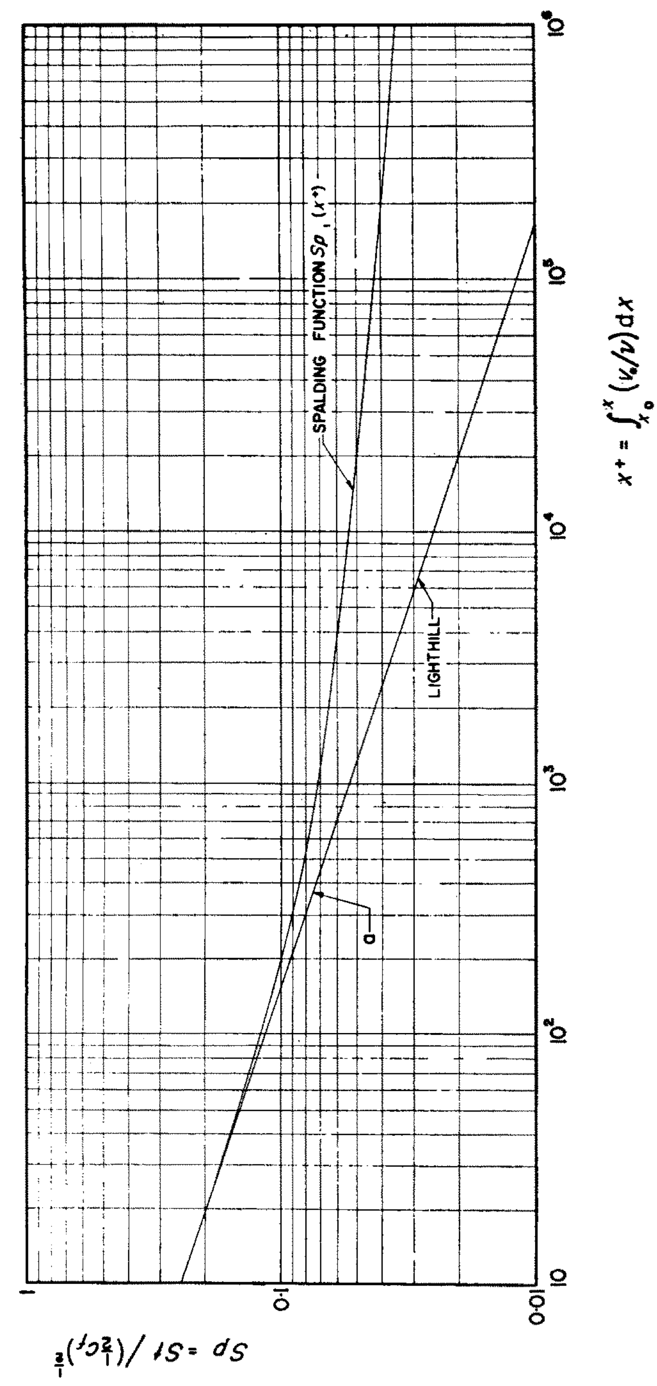

FIG. 2. The Spalding function $Sp_1(x^+)$ for $Pr = Pr_t = 1$; a—Lighthill's asymptotic solution from equation (9b).

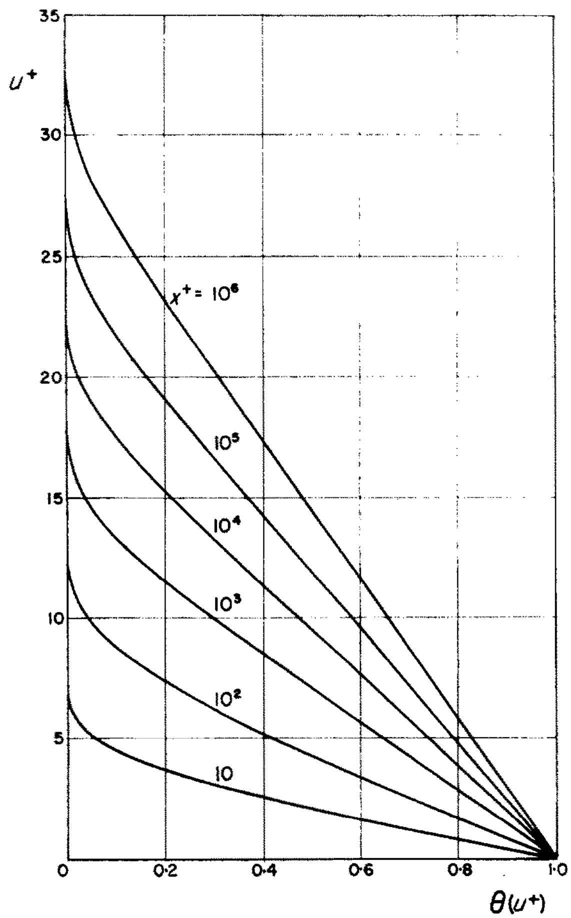

FIG. 3. Temperature profiles $\theta(u^+)$ for $x^+ = 10, 10^2, 10^3$ $10^4, 10^5, 10^6$.

and for the temperature profile:

$u^+ =$	Full steps	Halved steps
0·4	0·9408	0·9407
0·8	0·8815	0·8815
1·2	0·8224	0·8224
1·6	0·7635	0·7635
2·0	0·7049	0·7049

This was considered extremely satisfactory.

Having performed these tests, it became possible to carry out the calculation. Since it was desired to carry the tabulation to $x^+ = 10^6$, it became necessary to increase the steps as the calculation progressed in order to keep its duration within reasonable limits. This did not impair the accuracy of the calculation, since the temperature profile became progressively less curved and larger steps could be tolerated. The steps were chosen as follows:

Interval x^+	Δx^+	Δu^+	Largest φ
10 to 10^2	0·02	0·4	0·16
10^2 to 10^3	0·1	0·8	0·10
10^3 to 10^4	0·3	0·8	0·29
10^4 to 10^5	2·0	1·6	0·24
10^5 to 10^6	20·0	3·2	0·27

The results of the calculation are given in three forms. Table 1 contains the values of the

Table 1. The Spalding function $Sp_1(x^+)$ for $Pr = 1$

x^+	$Sp_1(x^+)$	x^+	$Sp_1(x^+)$
0	∞		
1	0.5384	1×10^3	0·07151
2	0·4273	2	0·06443
3	0·3733	3	0·06101
4	0·3391	4	0·05883
5	0·3148	5	0·05725
6	0·2963	6	0·05603
7	0·2814	7	0·05504
8	0·2692	8	0·05421
9	0·2588	9	0·05350
10	0·2498	1×10^4	0·05288
20	0·1987	2	0·04908
30	0·1742	3	0·04712
40	0·1589	4	0·04581
50	0·1481	5	0·04484
60	0·1399	6	0·04407
70	0·1335	7	0·04345
80	0·1282	8	0·04291
90	0·1238	9	0·04245
1×10^2	0·1200	1×10^5	0·04205
2	0·09929	2	0·03943
3	0·09012	3	0·03810
4	0·08466	4	0·03720
5	0·08092	5	0·03652
6	0·07815	6	0·03598
7	0·07598	7	0·03553
8	0·07423	8	0·03516
9	0·07276	9	0·03484
10^3	0·07151	10^6	0·03456

Spalding function Sp_1 from $x^+ = 0$ to $x^+ = 10^6$, and it is believed that these are accurate to three significant figures. A logarithimic plot of this function is given in Fig. 2, and Fig. 3 contains graphs of the temperature profiles $\theta(u^+)$ for selected values of x^+.

4. THE COEFFICIENT OF SKIN FRICTION ON A FLAT PLATE

The preceding calculations can be utilized for the computation of the skin friction coefficient on a flat plate in the presence of a turbulent boundary layer, assumed tripped at the leading edge. Such calculations have been performed by several investigators for different laws of the wall, as is well known [10], and they prove to be very cumbersome, even on the simplest assumptions. The difficulties are due, in essence, to the fact that any empirical law of the wall constitutes an expression for the turbulent viscosity μ_t which contains the skin friction. Thus the computation of the skin friction coefficient itself leads to a differential equation whose solution is troublesome, even if straightforward [11]. With the present results to hand, this can be done expeditiously if it is recalled that for $Pr = Pr_t = 1$ and along a flat plate the boundary layer profile must be similar to the temperature profile. This similarity, as is well known, can be reduced to the statement that

$$St = \tfrac{1}{2}c_f. \tag{11}$$

Combining the preceding statement with equation (8a), we obtain

$$Sp_1(x^+) = (\tfrac{1}{2}c_f)^{1/2} \tag{12}$$

which is valid *irrespective* of the Prandtl number, but *only* for a flat plate. Since x^+ depends on c_f, equation (12) is really a differential equation for c_f. That this is so can be seen by noting that

$$x^+ = \frac{\bar{U}_\infty}{\nu}\int_0^x (\tfrac{1}{2}c_f)^{1/2}\,\mathrm{d}\xi$$

where ξ is a dummy variable of integration. Hence

$$\frac{\mathrm{d}x^+}{\mathrm{d}x} = \frac{\bar{U}_\infty}{\nu}(\tfrac{1}{2}c_f)^{1/2}.$$

Introducing this into equation (12), we find that

$$\frac{U_\infty \mathrm{d}x}{\nu} = \frac{\mathrm{d}x^+}{Sp_1(x^+)} \tag{13}$$

which can be integrated at once. Introducing the length Reynolds number,

$$Re_x = \frac{U_\infty x}{\nu}$$

it is easy to show that

$$Re_x = \int_0^{x^+} \frac{\mathrm{d}\xi}{Sp_1(\xi)} \tag{14}$$

where the integration involves the Spalding function $Sp_1(x^+)$ expressed in terms of x^+, ξ denoting the dummy variable of integration. Equation (12) shows that $(\frac{1}{2}c_f)^{1/2}$ is simply equal to $Sp_1(x^+)$, and equation (14) yields the value of the length Reynolds number which corresponds to the coefficient of skin friction in question. Hence, in order to calculate the relation $c_f(Re_x)$ it is sufficient to perform one integration, equation (14), and some very simple additional computations whose nature is evident.

The integration indicated in equation (14) was performed automatically together with the calculations described earlier by the use of Simpson's rule. The result is given in Table 2.

It must be realized that, actually, the results quoted in Table 2 do not constitute a new, or improved evaluation of the skin friction coefficient for turbulent boundary layers. Rather, they should be regarded as a means of subjecting Spalding's theory of heat transfer into turbulent boundary layers to a test of validity. This involves two items. The first concerns Spalding's law of the wall, equation (5), together with the numerical constants (5b), and is of minor importance, since the direct comparison given in [4] is adequate for the purpose. The second concerns the approximation used for ϵ^+, and requires some elaboration.

The skin friction in a turbulent boundary layer is given by the equation

$$\tau = (\mu + \mu_t)\frac{\partial u}{\partial y}$$

which transforms to

$$\frac{\tau}{\tau_w} = \epsilon^+ \frac{\mathrm{d}u^+}{\mathrm{d}y^+}$$

instead of the expression given in equation (3a), and used in the theory. To a first approximation, the heat transfer problem can be solved by putting, provisionally

$$\frac{\tau}{\tau_w} = 1,$$

Table 2. Relation between length Reynolds number, equation (14), *and local skin-friction coefficient, equation* (12)

x^+	$(\frac{1}{2}c_f)^{1/2} = Sp$	c_f	Re_x	c_f equation (15)	c_f equation (16)
10^3	0·0715	0·0102	$11{\cdot}6 \times 10^3$	0·00977	0·00988
10^4	0·0529	0·00559	$16{\cdot}8 \times 10^4$	0·00525	0·00514
10^5	0·0420	0·00354	$21{\cdot}7 \times 10^5$	0·00328	0·00313
10^6	0·0346	0·00239	$26{\cdot}7 \times 10^6$	0·00224	0·00208

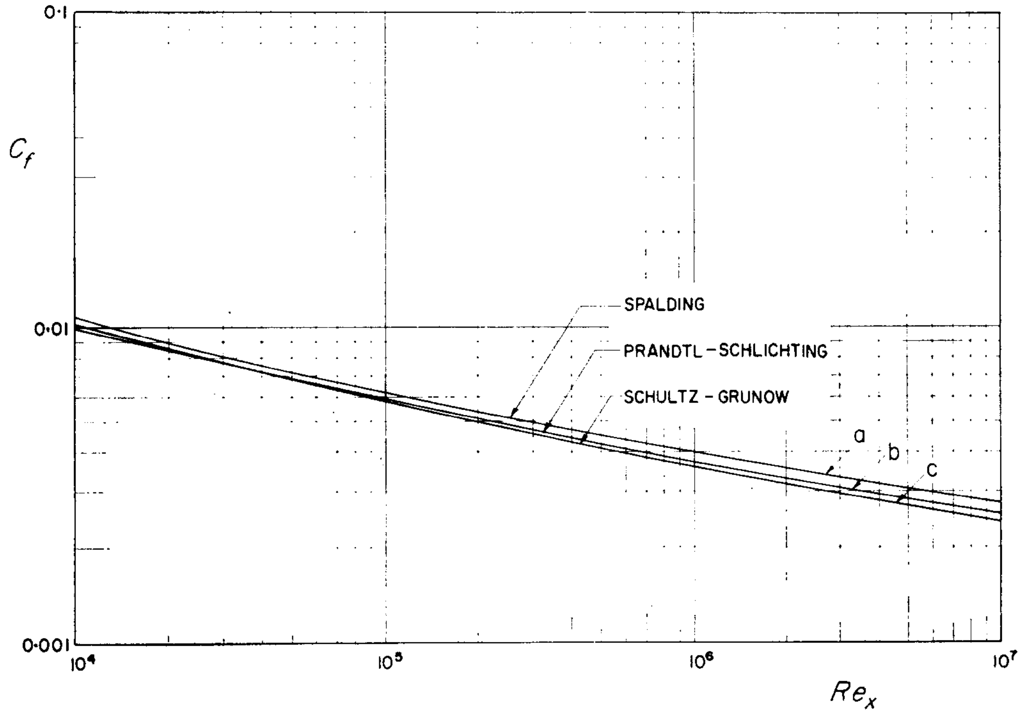

FIG. 4. Comparison between the values of the local skin-friction coefficient.
a—Spalding's law of the wall,
b—Prandtl–Schlichting, equation (15),
c—Schultz–Grunow, equation (16).

and then iterating on the solution. The application of this scheme to the computation of skin friction allows us to judge whether the iteration is required by direct comparison with experiment. A comparison between the values given in Table 2, and the well-known formulae due to Prandtl and Schlichting and Schultz–Grunow [12], respectively, namely

$$c_f = (2 \log Re_x - 0{\cdot}65)^{-2{\cdot}3}, \quad \text{(Prandtl–Schlichting)} \quad (15)$$

$$c_f = 0{\cdot}370 (\log Re_x)^{-2{\cdot}584}, \quad \text{(Schultz–Grunow)} \quad (16)$$

is contained in the table itself and in Fig. 4. Since the coefficients in equation (15) have been fitted to provide agreement with experimental data, the present comparison is equivalent to comparing the result of the integration with experiment subject to the above simplification.

The preceding comparison shows that Spalding's theory leads to consistently higher values of c_f, suggesting that most probably a second approximation might be useful. In view of the very large machine time involved, this was not done at the present time in the conviction that the approximation is adequate for heat transfer calculations. On the basis of auxiliary manual calculations, too tedious to repeat here, it would seem that the omission of the term $-(\frac{1}{24})\,(\kappa u^+)^4$ in equation (5) will remove this slight discrepancy.

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12. See Ref. 10, pp. 540 and 542.

Résumé—Cet article présente les détails du calcul numérique de la fonction, dite de Spalding, qui intervient dans la théorie du transport de chaleur à travers les couches limites turbulentes. Ce calcul se limite au cas d'un numbre de Prandtl égal à l'unité, quand on peut faire cette hypothèse. La méthode utilisée est une modification du schème d'intégration numérique en fonction du temps de Schmidt de l'equation de la conduction thermique unidimensionnelle de Fourier. Dans le cas traité, le terme qui représente la diffusivité thermique dépend de l'analogue de la coordonnée espace et entraîne une singularité à l'origine. Cette indétermination se lève en tenant compte du fait que les profils de température doivent avoir une courbure nulle à la paroi. L'analogie de Reynolds pour les couches limites turbulentes montre que l'on peut utiliser les calculs précédents pour établir une relation entre le coefficient de frottement à la paroi et le nombre de Reynolds (rapporté à la longueur), dans le cas d'une plaque plane léchée par une couche limite turbulente. L'intégration est très simple et fournit le moyen d'apprécier la précision de la théorie. Tout ceci est raisonablement satisfaisant.

Zusammenfassung—Es wird die numerische Berechnung der sogenannten Spaldingfunktion, wie sie in der Theorie des Wärmeübergangs in turbulenten Grenzschichten auftritt, im einzelnen beschrieben. Diese Berechnung beschränkt sich auf den Fall $Pr = 1$ und die Annahme, dass eine konstante turbulente Prandtlzahl von der Grösse 1 vorliegt. Die Rechnung wurde nach einem abgewandelten Schema von Schmidt durchgeführt, das die Integration der zeitabhängigen, eindimensionalen Fouriergleichung der Wärmeleitung ermöglicht. Im vorliegenden Fall hängt der Ausdruck, welcher der Temperaturleitzahl entspricht, von der Analogie der Raumkoordinaten ab und liefert am Ursprung eine singulare Gleichung. Diese Singularität wird umgangen, wenn man beachtet, dass die Temperaturprofile an der Wand eine Krümmung Null aufweisen müssen. Mit Hilfe der Reynoldsanalogie für turbulente Grenzschichten wird gezeigt, dass aus früheren Berechnungen eine Beziehung zwischen dem Koeffizienten der Oberflächenreibung und der Reynoldszahl, bezogen auf die Länge einer ebenen, von einer turbulenten Grenzschicht benetzten Platte, erhalten werden kann. Die Integration ist sehr einfach und gestattet, die Genauigkeit von Spaldings Theorie abzuschätzen. Dies erweist sich als genügend zufriedenstellend.

Аннотация—Описывается численный метод определения так называемой функции Сполдинга, применяемой в теории теплопереноса при турбулентном пограничном слое. Решение ограничивается случаем $Pr = 1$ и предположением, что турбулентное число Прандтля не превышает 1. Вычисление проводилось по модернизированному методу Шмидта, позволяющему проводить интегрирование одномерного уравнения Фурье. В данном случае выражение, соответствующее коэффициенту температуропроводности, зависит от подобия пространственных координат и определяется из решения сингулярного уравнения. Эту сингулярность можно обойти, если принять во внимание, что температурные профили на стенке имеют кривизну, равную нулю. С помощью аналогии Рейнольдса для турбулентных пограничных слоев показано, что из ранее проделанных вычислений можно получить соотношение между коэффициентом поверхностного трения и числом Рейнольдса, отнесенного к длине плоской пластины при наличии турбулентного слоя. Операция интегрирования очень простая и позволяет оценить точность закона Сполдинга на стенке. Закон оказывается достаточно точным.